Coordinate Geometry

solved exercises organized by skill level

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1 Introduction

Dear Reader,

These notes are related to the analytic geometry in plane and space, as currently studied at the scientific high school. I started by listing a range of skills for my students and for me. I then felt the need to complete this list with a brief theoretical indication and one or more examples relating to the individual items. These notes are designed especially for the students most in difficulty. If some passage seems to be carried out in a too extensive and detailed way, be patient: the best and most capable will understand the same, but we will not leave behind the less good.

Many exercises were written with the first numbers that came to mind, without looking for more pleasant solutions. These are not necessarily exercises to be given directly in the tests. Some fraction or obnoxious root can appear quietly: the world is not made only of whole numbers.

These notes are a support and complement to normal school textbooks.

I hope that what is reported in this work is if not helpful at least not harmful. To improve what is written and highlight any error do not hesitate to write to me. I apologize for my bad English.

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1.1 Licenza e Copyright

1.2 License and Copyright

δωρεὰν ἐλάβετε, δωρεὰν δότε (Mt. 7.8)

1.3 Credits

We thank those who have had the patience to read these pages and to report errors of various kinds. Especially:

Federico Belvisi, Jacopo Solinas, Anna Melis.

... So few find something to correct?

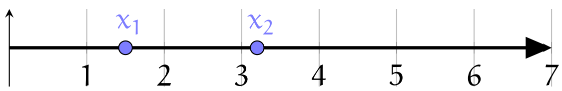
Part I

Rette e vettori

2 Points and vectors

2.1 Distance between two points on a straight line

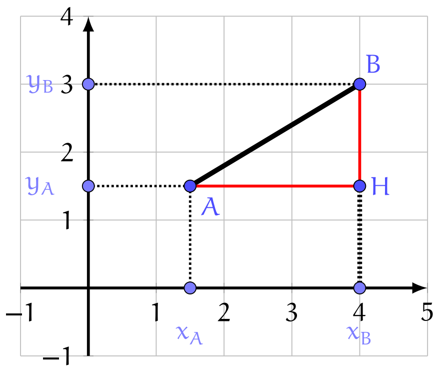
The distance between two points on a line is $d_{12} = |x_2 - x_1|$.



2.2 Distance between two points in the plane

The distance between a point $A(x_A, y_A)$ and point $B(x_B, y_B)$ is directly found by the following formula, obtained by applying the Pythagorean theorem to the triangle ABH:

$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \quad (2.1)$$



Obviously the distance between A and B is equal to the distance between B and A. Similarly, the order in which the points appear in the formula does not matter.

Esercizio 1 *Find the distance between points* $A(-3, 2)$ *e* $B(-2, -1)$.

$$d_{AB} = \sqrt{(-2-(-3))^2 + (-1-2)^2} = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10} \quad (2.2)$$

2.3 Sum and difference between two vectors

A vector $\vec{v}$ in space can be identified by the coordinates of its tip when applied to the origin of the Cartesian axes. These coordinates also represent the components of the vector with respect to the coordinate axes.

$$\vec{v} \equiv (v_x, v_y) \equiv v_x\vec{i} + v_y\vec{j} \tag{2.3}$$

The vectors $\vec{i}$ and $\vec{j}$ are unit vectors (versors) parallel to the coordinate axes.

The *algebraic sum* $\vec{c}$ (sum or difference) between two vectors $\vec{a}$ and $\vec{b}$ is a vector whose components are the algebraic sum of the respective components of $\vec{a}$ and $\vec{b}$.

$$\begin{aligned} \vec{c} &= \vec{a} + \vec{b} \\ (c_x, c_y) &= (a_x + b_x, a_y + b_y) \\ c_x\vec{i} + c_y\vec{j} &= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \vec{d} &= \vec{a} - \vec{b} \\ (d_x, d_y) &= (a_x - b_x, a_y - b_y) \\ d_x\vec{i} + d_y\vec{j} &= (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} \end{aligned} \tag{2.5}$$

Esercizio 2 *Find the sum* $\vec{c}$ *and the difference* $\vec{d}$ *between the vectors* $\vec{a} \equiv (-7, 6)$ *and* $\vec{b} \equiv (-3, -1)$.

We immediately apply what was written before:

$$\vec{c} \equiv (-7 - 3, 6 - 1) = (-10, 5) \tag{2.6}$$

$$\vec{d} \equiv (-7 - (-3), 6 - (-1)) = (-4, 7) \tag{2.7}$$

2.4 Vector between two points

Given two points A and B in space we can derive the vector $\overrightarrow{BA}$ which has for extremes the two points as the difference vector between the one associated with the second point minus the first.

$$\begin{aligned} \overrightarrow{BA} &= \vec{B} - \vec{A} \\ (ba_x, ba_y) &= (b_x - a_x, b_y - a_y) \end{aligned} \tag{2.8}$$

The modulus of a vector does not depend on the order in which we take the two points, the direction does.

2.5 The modulus of a vector

The *modulus* of a vector $\vec{a}$ is a number that expresses the intensity or length of the vector.

From the formula of the distance between two points (the origin of the axes and the tip of the vector) we can write:

$$|\vec{a}| = \sqrt{(a_x)^2 + (a_y)^2} \tag{2.9}$$

2.6 Product of a scalar for a vector

The product of a scalar k for a vector $\vec{a}$ is a vector whose components are the product of the scalar k for the components of the vector $\vec{a}$.

$$k \cdot \vec{a} = (ka_x, ka_y) \tag{2.10}$$

2.7 Scalar product between two vectors and cosine of the angle inclusive

The *scalar product* between two vectors $\vec{a}$ and $\vec{b}$ is a number (one scalar) related to the value of the two vectors.

If $|\vec{a}|$ is the modulus of the vector $\vec{a}$, $|\vec{b}|$ the modulus of the vector $\vec{b}$ and α the angle formed by them then we can define their scalar product as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\alpha \tag{2.11}$$

Or, taking the components of the two vectors:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \tag{2.12}$$

In a monometric Cartesian coordinate system the two definitions lead to the same result. The first definition is widely used in physics. In the context of analytic geometry, however, the latter is much more useful.

From the first definition, remembering that $\cos(90°) = 0$, we can say that, if two vectors are perpendicular, then their scalar product is zero.

Esercizio 3 *Find the scalar product* $\vec{a} \equiv (-7, 6)$ *e* $\vec{b} \equiv (-3, -1)$.

Immediately we write:

$$p = -7 \cdot (-3) + 6 \cdot (-1) = 15 \tag{2.13}$$

2.8 Find the perimeter and area of a polygon given the vertices

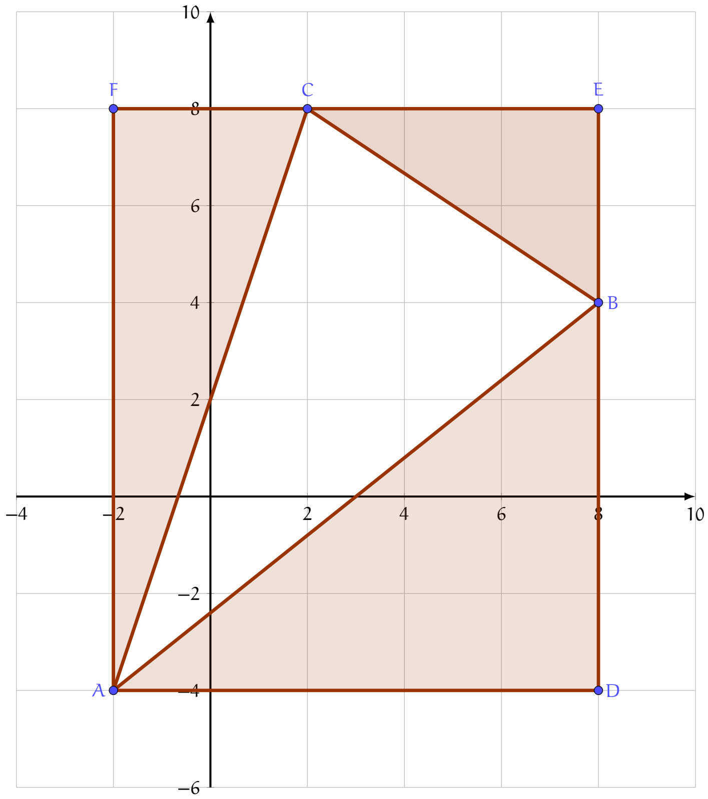
To find the *perimeter of a polygon* we add the length of all its sides. We determine the length of one side with the formula of the distance between two points.

To find the *area of a polygon* we decompose the polygon into squares, rectangles and triangles of base and known height and add up their area.

In particular, for the textitarea of a generic triangle, we construct the rectangle circumscribed to it and decompose the figure into right triangles. The area sought is the difference between the area of the rectangle and that of the right triangles outside our triangle.

Esercizio 4 *Find the area and perimeter of the triangle of vertices* $A(-2,-4)$, $B(8,4)$ *e* $C(2,8)$.

We report the triangle in a graph to see if the base and height are immediately visible and determinable. If, as in our case, this is not possible then we also draw the rectangle circumscribed to the triangle and highlight the right triangles that surround our triangle.



The area of the rectangle is $A_r = b \cdot h = 5 \cdot 6 = 30$.
The area of the triangle $A\widehat{C}F$ è $A_1 = \dfrac{b \cdot h}{2} = \dfrac{2 \cdot 6}{2} = 6$.
The area of the triangle $B\widehat{E}C$ è $A_2 = \dfrac{b \cdot h}{2} = \dfrac{3 \cdot 2}{2} = 3$.
The area of the triangle $A\widehat{D}B$ è $A_3 = \dfrac{b \cdot h}{2} = \dfrac{5 \cdot 4}{2} = 10$.
Finally, the triangle area $A\widehat{B}C$ è $A = A_r - A_1 - A_2 - A_3 = 30 - 6 - 3 - 10 = 11$.

To find the perimeter we determine the measurements of the sides:

$$\begin{aligned}\overline{AB} &= \sqrt{(2-(-2))^2+(8-(-4))^2} = \sqrt{4^2+12^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}\\ \overline{BC} &= \sqrt{(8-2)^2+(4-8)^2} = \sqrt{6^2+4^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}\\ \overline{AC} &= \sqrt{(8-(-2))^2+(4-(-4))^2} = \sqrt{10^2+8^2} = \sqrt{100+64} = \sqrt{164} = 2\sqrt{41}\end{aligned} \tag{2.14}$$

The perimeter is $p = \overline{AB} + \overline{BC} + \overline{AC} = 4\sqrt{10} + 2\sqrt{13} + 2\sqrt{41}$

2.9 Midpoint of a segment: definition and formula

The midpoint of a segment is the point of the segment equidistant from its extremes.
If the segment has for extremes the point $A(x_A, y_A)$ and the point $B(x_B, y_B)$ then the coordinates of the midpoint P_m are:

$$P_m(x_m, y_m) = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right) \tag{2.15}$$

2.10 Centroid of a polygon: definition and formula

The centroid of a polygon is a point whose coordinates are the arithmetic mean of the coordinates of the vertices of the polygon. If the polygon has n vertices then:

$$G(x_G, y_G) = \left(\frac{x_1 + x_2 + \dots + x_n}{n}, \frac{y_1 + y_2 + \dots + y_n}{n}\right) \tag{2.16}$$

Esercizio 5 *Find the centroid of the triangle of vertices* $A(2, -4)$, $B(-3, 5)$, $C(6, 7)$

Immediately we write:

$$G(x_G, y_G) = \left(\frac{2-3+6}{3}, \frac{-4+5+7}{3}\right) = \left(\frac{5}{3}, \frac{8}{3}\right) \tag{2.17}$$

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